ODD NUMBER AND TRAPEZOIDAL NUMBER

MASANORI ANDO

Abstract. In this paper, we give a bijective proof of the reduced lecture hall partition theorem. It is possible to extend this bijection in lecture hall partition theorem. And refined versions of each theorems are also presented.

1. Introduction

$$\frac{1}{1}=1,\ \frac{3\cdot 2}{1\cdot 3}=2,\ \frac{6\cdot 5\cdot 3}{1\cdot 3\cdot 5}=6,\ \frac{10\cdot 9\cdot 7\cdot 4}{1\cdot 3\cdot 5\cdot 7}=24,\ \frac{15\cdot 14\cdot 12\cdot 9\cdot 5}{1\cdot 3\cdot 5\cdot 7\cdot 9}=120.$$

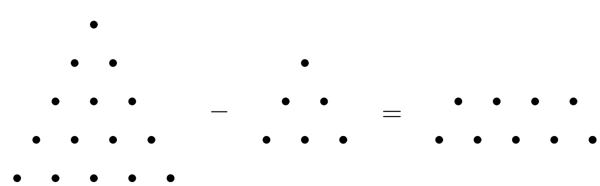
There is the product of odd numbers in denominator. The numbers in numerator are the differences between two triangular numbers. And the calculation results are the factorial numbers. The proof of this result is not so difficult. We consider what generating function is the q-analogue of this calculation.

2. Odd number and trapezoidal number

Theorem 2.1. *For any positive integer n,*

$$\prod_{k=1}^{n}\frac{\binom{n+1}{2}-\binom{k}{2}}{2k-1}=n!.$$

Figure.



Number n means bottom length of trapezoid as it's showed in figure. It's natural to classify it by top length. However we won't do that.

For positive integer k, n, we define trapezoidal number

$$\Diamond_{n,k}:=\begin{cases} n+(n-1)+\cdots+(n-2k+2) & ({}^{\forall}k\le \frac{n}{2}) \\ n+(n-1)+\cdots+(2k-n-1) & ({}^{\forall}k> \frac{n}{2}) \end{cases}$$

Lemma 2.2. *For positive integer $k, n, k \le n$,*

$$\frac{\Diamond_{n,k}}{2k-1}=n-k+1.$$

2010 *Mathematics Subject Classification.* 05A15, 05A17.
Key words and phrases. integer partition, lecture hall partition.

Proof. Let calculate the sum of the number sequence.

$$
\begin{aligned}
&{}^{\forall}k \le \frac{n}{2},\\
&\quad \Diamond_{n,k} = n + (n-1) + \cdots + (n-2k+2)\\
&\quad = \frac{1}{2}(n + (n-2k+2)) \times (n - (n-2k+2) + 1)\\
&\quad = \frac{1}{2}(2n-2k+2) \times (2k-1)\\
&\quad = (n-k+1) \times (2k-1),\\
&{}^{\forall}k > \frac{n}{2},\\
&\quad \Diamond_{n,k} = n + (n-1) + \cdots + (2k-n-1)\\
&\quad = \frac{1}{2}(n + (2k-n-1)) \times (n - (2k-n-1) + 1)\\
&\quad = \frac{1}{2}(2k-1) \times (2n-2k+2)\\
&\quad = (2k-1) \times (n-k+1).
\end{aligned}
$$

□

We permute numerator numbers by a "Skip" like next example. Then Theorem is obvious.

Example. For $n = 6$,

$$
\frac{21 \cdot 20 \cdot 18 \cdot 15 \cdot 11 \cdot 6}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} = \frac{6 \cdot 15 \cdot 20 \cdot 21 \cdot 18 \cdot 11}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!.
$$

We consider the q-analogue of theorem 2.1. By the previous reduction,

$$
\prod_{k=1}^{n} \frac{1 - q^{\binom{n}{2} - \binom{k-1}{2}}}{1 - q^{2k-1}} = \prod_{k=1}^{n} \sum_{i=0}^{n-k} q^{i(2k-1)}.
$$

This is the generating function of the odd partitions which that the number of $2k-1$ part is less than or equal to $n-k$.

3. Reduced lecture hall partition theorem

Let n be a positive integer. A partition λ of n is an integer sequence

$$
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)
$$

satisfying $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_\ell > 0$ and $\sum_{i=1}^{\ell} \lambda_i = n$. We call $\ell(\lambda) := \ell$ the length of λ, and each λ_i a part of λ. We let $\mathcal{P}$ and $\mathcal{P}(n)$ denote the set of partitions and the set of partitions of n. For a partition λ, we let $m_i(\lambda)$ denote the multiplicity of i as its part. $(1^{m_1(\lambda)} 2^{m_2(\lambda)} \ldots)$ is another representation of λ. We define addition and subtraction of partition each two pattern.

$$
\begin{aligned}
\lambda + \mu &:= (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots,),\\
\lambda - \mu &:= (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots,),\\
\lambda \sqcup \mu &:= (1^{m_1(\lambda)+m_1(\mu)} 2^{m_2(\lambda)+m_2(\mu)} \ldots),\\
\lambda \setminus \mu &:= (1^{m_1(\lambda)-m_1(\mu)} 2^{m_2(\lambda)-m_2(\mu)} \ldots).
\end{aligned}
$$

A partition $\lambda = (1^{m_1}2^{m_2}3^{m_3}\dots)$ is said to be odd if $m_{2i} = 0$ for all i. We denote $\mathcal{OP}$ and $\mathcal{OP}(n)$ the set of odd partitions and the set of odd partitions of n.

For positive integer N, partition μ is called lecture hall partition when μ satisfies

$$\frac{\mu_1}{N} \geq \frac{\mu_2}{N-1} \geq \dots \geq \frac{\mu_{\ell(\mu)}}{N-\ell(\mu)+1}.$$

We call N the width of lecture hall. We denote $\mathcal{L}_N$ the set of lecture hall partitions of width N.

Example. $(4,3,1) \notin \mathcal{L}_3$ because $\frac{4}{3} > \frac{3}{1}$. $(4,3,1) \in \mathcal{L}_4$ because $\frac{4}{4} \leq \frac{3}{3} \leq \frac{1}{2} \leq \frac{0}{1}$.

A lecture hall partition $\mu = (\mu_1, \mu_2, \dots, \mu_{\ell(\mu)})$ is said to be reduced if

$$^{\forall}i, (\mu_1, \mu_2, \dots (\mu_i - (N-i+1)), \dots, \mu_{\ell_\mu}) \notin \mathcal{L}_N.$$

We let $\mathcal{RL}_N$ denote the set of reduced lecture hall partitions of width N.

Example. For $N = 3$.

$$\mathcal{RL}_3 = \{\emptyset, (1), (2), (2,1), (3,1), (4,1))\}.$$

Here $(5,1) \in \mathcal{L}_N$, but $(5,1) \notin \mathcal{RL}_N$. Because $\frac{5}{3} - 1 > \frac{1}{2}$.

For positive integer $k, N, (k \leq N)$, we denote that,

$$[\Diamond]_{N,k} := \begin{cases} (N, N-1, \dots, N-2k+2) & (^{\forall}k \leq \frac{N}{2}) \\ (N, N-1, \cdots, 2k-N-1) & (^{\forall}k > \frac{N}{2}) \end{cases}.$$

For any $[\Diamond]_{N,k} = \mu = (\mu_1, \mu_2, \dots, \mu_\ell)$, obviously

$$\frac{\mu_1}{N} = \frac{\mu_2}{N-1} = \dots = \frac{\mu_\ell}{N-\ell+1} = 1$$

Then $[\Diamond]_{N,k}$ is lecture hall partition of width N but not reduced.

Proposition 3.1. *For any positive integer N,*

$$\sharp\mathcal{RL}_N = N!.$$

Proof. By the definition,

$$\mu = (\mu_1, \mu_2, \dots, \mu_{\ell(\mu)}) \in \mathcal{RL}_N \Rightarrow (\mu_2, \dots, \mu_{\ell(\mu)}) \in \mathcal{RL}_{N-1}.$$

When $\mu_2, \dots, \mu_{\ell(\mu)}$ are fixed, there are N pattern m_1 such that $\mu \in \mathcal{RL}_N$. Then it was proved by mathematical induction. □

We also ready reduced odd partition. Let N be a positive integer. We denote,

$$\mathcal{OP}_N := \{\lambda \in \mathcal{OP} \mid {}^{\forall}i > 2N, m_i = 0\},$$

$$R\mathcal{OP}_N := \{\lambda \in \mathcal{OP}_N \mid {}^{\forall}k \leq N, m_{2k-1} \leq N-k\}.$$

We call them the set of N-party odd partitions, the set of N-party reduced odd partitions. The next proposition is obvious from definition.

Proposition 3.2. *For any positive integer N,*

$$\sharp\mathcal{OP}_N = N!.$$

Then $\sharp\mathcal{RL}_N = \sharp\mathcal{ROP}_N$. We also prove the orders of these sets are equal in every size n.

Theorem 3.3 (Reduced Lecture Hall Partition Theorem)**.** *For any positive integer N,*

$$\sum_{\mu\in\mathcal{RL}_N} q^{|\mu|} = \sum_{\lambda\in R\mathcal{OP}_N} q^{|\lambda|}.$$

For proof of this theorem, we construct the map from $\mathcal{OP}_N$ to $\mathcal{L}_N$

$$\begin{array}{cccccc} \Phi_N: & \mathcal{OP}_N & \longrightarrow & \mathcal{L}_N & & \\ & \cup & & \cup & & \\ & \lambda & \longmapsto & \mu & \text{s.t.}|\lambda|=|\mu| \end{array}.$$

And we prove this map is bijection. We remark beforehand the map Φ_N is equivalent with [2]. However, the using sets become finite by reduction. Then the proof becomes easy. And we also give bijective proof of lecture hall partition theorem by Φ_N.

Fist, let consider the case $N=\infty$, $\Phi := \Phi_\infty$. For $\lambda = ((2k-1)^m)$ which consists of a kind of part $2k-1$, we define

$$\Phi((2k-1)^m) := [\Diamond]_{k+m-1,k}.$$

And we put $A_{i,k}$ the increase from $\Phi((2k-1)^{i-1})$ to $\Phi((2k-1)^i)$. [1]

$$A_{i,k} := \Phi((2k-1)^i) - \Phi((2k-1)^{i-1}) = [\Diamond]_{k+i-1,k} - [\Diamond]_{k+i-2,k}.$$

Example. Let $2k-1=5$,

$$\Phi(5^0) = [\Diamond]_{2,3} = \emptyset, \Phi(5) = [\Diamond]_{3,3} = (3,2), \Phi(5^2) = [\Diamond]_{4,3} = (4,3,2,1),$$
$$\Phi(5^3) = [\Diamond]_{5,3} = (5,4,3,2,1), \Phi(5^4) = [\Diamond]_{6,3} = (6,5,4,3,2), \ldots.$$

Then

$$A_{1,3} = (3,2), A_{2,3} = (1,1,2,1), A_{3,3} = (1,1,1,1,1), A_{4,3} = (1,1,1,1,1), \ldots.$$

We grow up lecture hall partition by combination of this increase A. We define next counter $I(\lambda) = [I_1(\lambda), I_2(\lambda), \ldots, I_N(\lambda)]$ to select increase A. The default of counter is $I(\lambda) = [0,0,\ldots 0]$. For $^\forall\lambda \in \mathcal{OP}_N, {}^\forall 2k-1 \le \lambda_{\ell(\lambda)}$,

$$\Phi_N(\lambda \sqcup (2k-1)^1) := \Phi_N(\lambda) + A_{i(\lambda),k}.$$

Here, $i(\lambda) := \min\{j \mid I_j(\lambda) = 0\}$.

$$\begin{aligned} & I(\lambda \sqcup (2k-1)^1) \\ = \;& [I_1(\lambda)-1, \ldots, I_{i(\lambda)-1}(\lambda)-1, N-k-i(\lambda)+1, I_{i(\lambda)+1}(\lambda), \ldots, I_N(\lambda)]. \end{aligned}$$

Example. For $N=7$, $1^4 3^2 7^3 911 \in \mathcal{OP}_7$.

$\Phi(11) = \emptyset + A_{1,6} = (6,5)$,	$I(11) = [1,0,0,0,0,0,0]$,
$\Phi(911) = (6,5) + A_{2,5} = (7,6,4,3)$,	$I(911) = [0,1,0,0,0,0,0]$,
$\Phi(7911) = (7,6,4,3) + A_{1,4} = (11,9,4,3)$,	$I(7911) = [3,1,0,0,0,0,0]$,
$\Phi(7^2 911) = (11,9,4,3) + A_{3,4} = (12,10,5,4,2,1)$,	$I(7^2 911) = [2,0,1,0,0,0,0]$,
$\vdots$,	$\vdots$
$\Phi(1^4 3^2 7^3 911) = (20,13,9,6,2,1)$,	$I(1^4 3^2 7^3 911) = [1,3,2,0,1,1,0]$.

We research the properties of counter I. For $\lambda \in \mathcal{OP}_N, \mu = \Phi_N(\lambda)$, We put $d_j(\lambda)$ the number of action j while we grow up μ.

[1] $A_{i,k}$ may not be a partition. But we applied same calculation "+", "−".

Example. For $\lambda = 1^4 3^2 7^3 911 \in \mathcal{OP}_7$ just before example,

$$d_1(\lambda) = 3, d_2(\lambda) = 3, d_3(\lambda) = 2.$$

Lemma 3.4. *Let $\lambda \in \mathcal{OP}_N, \mu = \Phi_N(\lambda)$, $2k-1 = \lambda_{\ell(\lambda)}$. Then,*

$$\forall j \le k, \qquad \mu_{2j-1} = (N-2j+2)d_j(\lambda) - I_j(\lambda) \tag{3.1}$$
$$\forall j < k, \qquad \mu_{2j} = (N-2j+1)d_j(\lambda) - I_j(\lambda). \tag{3.2}$$

Especially,

$$\forall j \le k, \qquad \mu_{2j-1} \equiv -I_j(\lambda) \pmod{N-2j+2} \tag{3.3}$$
$$\forall j < k, \qquad \mu_{2j} \equiv -I_j(\lambda) \pmod{N-2j+1}. \tag{3.4}$$

Proof. When default $\lambda = \mu = \emptyset, I(\emptyset) = [0, 0, \dots 0]$, both side equal 0. [2]

Let prove equation inductive. We put $\lambda' = \lambda \setminus (2k-1), \mu' = \Phi_N(\lambda')$. And we assume that λ' satisfies identity (3.1), (3.2). If $i(\lambda') > j$,

$$d_j(\lambda) = d_j(\lambda'), I_j(\lambda) = I_j(\lambda') - 1.$$

From definition of A,

$$\forall j \le k, \qquad \mu_{2j-1} = \mu'_{2j-1} + 1,$$
$$\forall j < k, \qquad \mu_{2j} = \mu'_{2j} + 1.$$

If $i(\lambda') = j$,

$$d_j(\lambda) = d_j(\lambda') + 1, I_j(\lambda) = I_j(\lambda') + N - k - j + 1,$$
$$\forall j \le k, \quad \mu_{2j-1} = \mu'_{2j-1} + k - j + 1, \mu_{2j} = \mu'_{2j} + k - j.$$

If $i(\lambda') < j$,

$$d_j(\lambda) = d_j(\lambda'), I_j(\lambda) = I_j(\lambda'), \mu_{2j-1} = \mu'_{2j-1}, \mu_{2j} = \mu'_{2j}.$$

Then λ satisfies equations. □

Lemma 3.5. *Let $\lambda \in \mathcal{ROP}_N, (I, \mu) = \Phi(\lambda)$. For all j,*

$$0 \le d_j(\lambda) - d_{j+1}(\lambda) \le 1$$

Proof. By the definition of I and Φ_N, first $A_{j,\bullet}$ is previous than first $A_{j+1,\bullet}$. When $d_j(\lambda) - d_{j+1(\lambda)} = 1$, $I_j(\lambda) > I_{j+1}(\lambda)$. [3] Then, next $A_{j+1,\bullet}$ is previous than $A_{j,\bullet}$. When $d_j(\lambda) - d_{j+1(\lambda)} = 0$, $I_j(\lambda) \le I_{j+1}(\lambda)$. Then, next $A_{j,\bullet}$ is previous than $A_{j+1,\bullet}$. □

Let $\lambda \in \mathcal{OP}_N$, $\mu = \Phi_N(\lambda) \in \mathcal{RL}_N$, $\lambda_\ell(\lambda) = 2k-1$. For $l \le k$, we put $\lambda' := \lambda \sqcup (2l-1)$, $\mu' := \Phi_N(\lambda')$. Because $\ell(A_{\bullet,l}) \le 2l-1$, μ_j equals μ'_j for all j greater than $2l-1$. Then,

$$(\mu'_{2l}, \mu'_{2l+1}, \dots, \mu'_{\ell(\mu')}) \in \mathcal{RL}_{N-(2l-1)}.$$

Therefore the possibility of failure of the inequalities of "reduced" and "lecture hall" exists in only μ'_js ($j \le 2l$). By the Lemma 3.5 and (3.1), (3.2) of Lemma 3.7,

$$\mu' \notin \mathcal{L}_N \Leftrightarrow \exists j < l, d_j(\lambda') - d_{j+1}(\lambda') = 0 \wedge I_j(\lambda') > I_{j+1}(\lambda').$$

[2] When $\lambda = \emptyset$, we regard k as $2N-1$ (unlimited).

[3] When $\lambda \notin \mathcal{ROP}_N$, there are case $I_j(\lambda) = I_{j+1}(\lambda) = 0$.

The right-hand side is false by the argument of the proof of Lemma 3.5. Then Φ_N is map from $\mathcal{OP}_N$ to $\mathcal{L}_N$. And,

$$\mu' \notin \mathcal{RL}_N \Leftrightarrow {}^{\exists} j < l, d_j(\lambda') - d_{j+1}(\lambda') = 1 \wedge I_j(\lambda') \leq I_{j+1}(\lambda').$$

The right-hand side is false when $\lambda' \in \mathcal{ROP}_N$. It follows the next proposition.

Proposition 3.6. *Let $\lambda \in \mathcal{OP}_N$. Then,*

$$\lambda \in \mathcal{ROP}_N \Longrightarrow \Phi_N(\lambda) \in \mathcal{RL}_N.$$

Lemma 3.7. *Let $\lambda \in \mathcal{OP}_N, 2k-1 \leq \lambda_{\ell(\lambda)}$. Then,*

$$\Phi_N(\lambda \sqcup (2k-1)^{N-k+1}) = \Phi_N(\lambda) + [\Diamond]_{N,k},$$

$$I(\lambda \sqcup (2k-1)^{N-k+1}) = I(\lambda).$$

Proof. For any $a \leq N-k+1$, we put $\lambda^{(a)} := \lambda \sqcup (2k-1)^a$. Because the smallest part of $\lambda^{(a)}$ is $2k-1$,

$$I_{N-k+1}(\lambda^{(a)}) = I_{N-k+2}(\lambda^{(a)}) = \ldots = I_N(\lambda^{(a)}) = 0.$$

Then $i(\lambda^{(a)}) \leq N-k$. We prove that

$$\{i(\lambda), i(\lambda^{(1)}), \ldots, i(\lambda^{(N-k)})\} = \{1, 2, \ldots, N-k+1\}.$$

Then,

$$\Phi_N(\lambda \sqcup (2k-1)^{N-k+1}) = \Phi_N(\lambda) + \sum_{a=0}^{N-k} A_{i(\lambda^a),k} = \sum_{j=1}^{N-k+1} A_{j,k} = \Phi_N(\lambda) + [\Diamond]_{N,k}.$$

First ,we assume that $i(\lambda^{(a)})$ is not equal to 1 for all a. Because $I_1(\lambda)$ is less than $N-k$ and $I_1(\lambda^{(a+1)}) = I_1(\lambda^{(a)}) - 1$, $I_1(\lambda^{(N-k+1)}) < 0$. It is incompatible. We put $i(\lambda^{(a_1)}) = 1$. Then $I_1(\lambda^{(a_1+1)})$ equals $N-k$. For all a bigger than a_1, $I_1(\lambda^{(a)}) > 0$. Therefore the a that $i(\lambda^{(a)}) = 1$ is only a_1.

Next, we assume that $i(\lambda^{(a)})$ is not equal to 2 for all a. Because $I_2(\lambda)$ is less than $N-k-1$ and $I_2(\lambda^{(a+1)}) = I_2(\lambda^{(a)}) - 1$ for all $a \neq a_1$, $I_2(\lambda^{(N-k+1)}) < 0$. It is compatible too. We put $i(\lambda^{(a_2)}) = 2$. Then $I_2(\lambda^{(a_1+1)})$ equals $N-k-1$. For all a bigger than a_2, $I_2(\lambda^{(a)}) > 0$. Therefore the a that $i(\lambda^{(a)}) = 2$ is only a_2.

$$\vdots$$

For all j less than $N-k+1$, the a that $i(\lambda^{(a)}) = j$ is only a_j. And we recall that $i(\lambda^{(a)}) \leq N-k+1$. Then the last one action is $A_{N-k+1,k}$.

Let fix $l \leq N-k+1$.

$$I_l(\lambda^{(a_j+1)}) = \begin{cases} I_l(\lambda^{(a_j)}) & (j < l) \\ I_l(\lambda^{(a_j)}) + N-k-l+1 & (j = l) \\ I_l(\lambda^{(a_j)}) - 1 & (j > l) \end{cases}.$$

Then,

$$I_l(\lambda^{(N-k+1)}) = I_l(\lambda) + (N-k-l+1) - 1 \times (N-k-l+1) = I_l(\lambda).$$

$\square$

Proposition 3.8. *Let $\lambda \in \mathcal{OP}_N$. Then,*

$$\Phi_N(\lambda \sqcup (2k-1)^{N-k+1}) = \Phi_N(\lambda) + [\Diamond]_{N,k}.$$

Therefore,

$$\lambda \notin \mathcal{ROP}_N \Longrightarrow \Phi_N(\lambda) \notin \mathcal{RL}_N.$$

Proof. By the Lemma 3.7,

$$2k-1 \leq \lambda_{\ell(\lambda)} \Rightarrow I(\lambda \sqcup (2k-1)^{N-k+1}) = I(\lambda).$$

Then the growths after that are not change. $\square$

Lemma 3.9. *Let $\lambda, \mu \in R\mathcal{OP}_N$, $\Phi_N(\lambda) = \Phi_N(\mu)$, $2k-1 = (\lambda \sqcup \mu)_{\ell(\lambda \sqcup \mu)}$. Then,*

$$^\forall l \leq k, \Phi_N(\lambda \sqcup (2l-1)) = \Phi_N(\mu \sqcup (2l-1)).$$

Proof. By $\Phi_N(\lambda) = \Phi_N(\mu)$ and (3.1),

$$^\forall j \leq k, I_j(\lambda) = I_j(\mu).$$

Then,

$$i(\lambda) \leq k \vee i(\mu) \leq k \Rightarrow i(\lambda) = i(\mu) \Rightarrow A_{i(\lambda),l} = A_{i(\mu),l}.$$

From definition of A,

$$i(\lambda) > k \wedge i(\mu) > k \Rightarrow A_{i(\lambda),l} = A_{i(\mu),l}.$$

Then,

$$\Phi_N(\lambda \sqcup (2l-1)) = \Phi_N(\mu \sqcup (2l-1)) = \Phi_N(\lambda) + A_{i(\lambda),l}.$$

$\square$

Proposition 3.10. *Φ_N is injection.*

Proof. Let $\lambda, \mu \in \mathcal{OP}_N, \lambda \neq \mu$. We assume that $\Phi_N(\lambda) = \Phi_N(\mu)$. And we put $2k-1 = (\lambda \sqcup \mu)_{\ell(\lambda \sqcup \mu)}$. If $m_{2k-1}(\lambda) = m_{2k-1}(\mu) =: m_{2k-1}$, we transform λ, μ as

$$\lambda \mapsto \lambda' := \lambda \setminus (2k-1)^{m_{2k-1}},$$
$$\mu \mapsto \mu' := \mu \setminus (2k-1)^{m_{2k-1}}.$$

Then,

$$\begin{aligned}\Phi_N(\lambda') &= \Phi_N(\lambda \sqcup (2k-1)^{N-k-m_{2k-1}+1}) - [\Diamond]_{N,k} \\ &= \Phi_N(\mu \sqcup (2k-1)^{N-k-m_{2k-1}+1}) - [\Diamond]_{N,k} = \Phi_N(\mu').\end{aligned}$$

Repeat this transform until the multiples of the smallest parts will be different and less than $2N-k$. Let $m_{2k-1}(\lambda) > m_{2k-1}(\mu)$. Then,

$$\lambda \sqcup (2k-1)^{N-k-m_{2k-1}(\lambda)} \notin \mathcal{ROP}_N, \mu \sqcup (2k-1)^{N-k-m_{2k-1}(\lambda)} \in \mathcal{ROP}_N.$$

Therefore,

$$\Phi_N(\lambda \sqcup (2k-1)^{N-k-m_{2k-1}(\lambda)}) \notin \mathcal{RL}_N, \Phi_N(\mu \sqcup (2k-1)^{N-k-m_{2k-1}(\lambda)}) \in \mathcal{RL}_N.$$

It is compatible. $\square$

Then Φ is injection between same order sets. Therefore Φ is bijection. We proved Theorem 3.3.

For any positive integer N,

$$\sum_{\lambda \in \mathcal{ROP}_N} q^{|\lambda|} = \sum_{\mu \in \mathcal{RL}_N} q^{|\mu|} = \prod_{k=1}^{N} \frac{1 - q^{\Diamond_{N,k}}}{1 - q^{2k-1}}.$$

4. Lecture hall partition theorem

We consider reduction of odd partition and lecture hall partition. First, for odd partition. Let $\lambda = (1^{m_1}2^{m_2}\ldots) \in \mathcal{OP}_N$. If $m_k > N-k$, we transform λ as

$$(1^{m_1}2^{m_2}\ldots k^{m_k}\ldots) \mapsto (1^{m_1}2^{m_2}\ldots k^{m_k-(N-k+1)}\ldots).$$

Repeat this transform until $m_k \leq N-k$ for all k. Then the resulting partition $[\lambda]$ is reduced odd partition.

Next, for lecture hall partition. Let $\mu = (\mu_1, \mu_2, \ldots) \in \mathcal{L}_N$. If $\frac{\mu_j}{N-j+1} - 1 \geq \frac{\mu_{j+1}}{N-j}$, then we transform μ as

$$\mu \mapsto \mu - (N, N-1, \ldots, N-j+1) = \begin{cases} \mu - [\Diamond]_{N,j} & (j : \text{odd}) \\ \mu - [\Diamond]_{N,2N-j+1} & (j : \text{even}) \end{cases}.$$

Repeat this transform until $\frac{\mu_j}{N-j+1} - 1 < \frac{\mu_{j+1}}{N-j}$ for all j. Then the resulting partition $[\mu]$ is reduced lecture hall partition.

By the Lemma 2.2, $|[\Diamond]_{N,k}| = |(2k-1)^{N-k+1}| = \Diamond_{N,k}$. Then,

$$\begin{aligned}
&\sum_{\lambda \in \mathcal{OP}_N} q^{|\lambda|} \\
=\ & \prod_{k=1}^{N} \sum_{j \geq 0} q^{j \times |(2k-1)^{N-k+1}|} \times \sum_{\lambda \in \mathcal{ROP}_N} q^{|\lambda|} \\
=\ & \prod_{k=1}^{N} \frac{1}{1-q^{\Diamond_{N,k}}} \times \prod_{k=1}^{N} \frac{1-q^{\Diamond_{N-k}}}{1-q^{2k-1}} \\
=\ & \prod_{k=1}^{N} \sum_{j \geq 0} q^{j \times |[\Diamond]_{N,k}|} \times \sum_{\mu \in \mathcal{RL}_N} q^{|\mu|} \\
=\ & \sum_{\mu \in \mathcal{L}_N} q^{|\mu|}.
\end{aligned}$$

Theorem 4.1 (Lecture Hall Partition Theorem [1]). *For any positive integer N,*

$$\sum_{\lambda \in \mathcal{OP}_N} q^{|\lambda|} = \sum_{\mu \in \mathcal{L}_N} q^{|\mu|} = \prod_{k=1}^{N} \frac{1}{1-q^{2k-1}}.$$

Same correspondence of $(2k-1)^{N-k+1}$ and $[\Diamond]_{N,k}$ was proved about a property of Φ_N. Then Φ_N is also bijection from $\mathcal{OP}_N$ to $\mathcal{L}_N$.

Last of this paper, we introduce refined versions of lecture hall partition theorems. Let $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}$, we denote the alternative size of λ

$$|\lambda|_a := \sum_{k=1}^{\ell(\lambda)} (-1)^{k+1} \lambda_k.$$

For trapezoid,

$$|[\Diamond]_{N,k}|_a = k.$$

Then,

$$|A_{N,k}|_a = |[\Diamond]_{N,k}|_a - |[\Diamond]_{N,k-1}|_a = 1.$$

Next theorems follow from the definition of Φ_N.

Theorem 4.2. *For any positive integer N,*

$$\sum_{\lambda \in \mathcal{OP}_N} t^{\ell(\lambda)} q^{|\lambda|} = \sum_{\mu \in \mathcal{L}_N} t^{|\mu|_a} q^{|\mu|} = \prod_{k=1}^{N} \frac{1}{1 - tq^{2k-1}}.$$

Theorem 4.3. *For any positive integer N,*

$$\sum_{\lambda \in \mathcal{ROP}_N} t^{\ell(\lambda)} q^{|\lambda|} = \sum_{\mu \in \mathcal{RL}_N} t^{|\mu|_a} q^{|\mu|} = \prod_{k=1}^{N} \frac{1 - t^{N-k+1} q^{\diamondsuit_{N,k}}}{1 - tq^{2k-1}}.$$

References

[1] M. Bousquet-Mélou and K. Eriksson, Lecture hall partitions, *Ramanujan J.* **1** (1997a) 101-110.

[2] A. J. Yee, On the combinatorics of lecture hall partitions, *Ramanujan J.* **5** (2001) 247-262.

Wakhok University, Hokkaido 097-0013, Japan

E-mail address: m-ando@wakhok.ac.jp